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On double-roll convection in a rotating magnetic system

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Electrically and thermally conducting inviscid fluid rotating about a vertical axis is confined between two horizontal plates maintained at different temperatures, the upper plate being the cooler. The fluid is permeated by a horizontal magnetic field that corotates with the fluid. In an earlier paper (Roberts & Stewartson 1974) the fluid is supposed to be in a state of near-marginal instability to convective overturning and the nonlinear evolution of single rolls is discussed. Inertial terms are neglected. However, if $q < 2$ and $\lambda < 2/3^{\frac{1}{2}}$, where q and λ may be defined by equation (2.3) below, the principle of the exchange of stabilities holds and there is also a degeneracy in the linear stability problem. There are now two distinct unstable rolls equally possible and their nonlinear interaction leads to a violation of the governing equations. This difficulty has already been noted by Taylor (1963) and it is resolved in this paper by adding a geostrophic motion (the Taylor shear) parallel to the magnetic field and by restoring the inertial terms in the governing equations. We consider particularly instabilities in which one roll predominates and find that, if λ is sufficiently small, each of the rolls that can occur is stable with respect to the other, i.e. an initially weak roll of the other type dies out relative to it. This means that we can expect the fluid motion to consist of single rolls at large times. On the other hand when λ is near $2/3^{\frac{1}{2}}$ both rolls are unstable with respect to the other. The Taylor shear does not then die out and the two rolls become comparable in magnitude and modify each other's structure. At intermediate values of λ one of the rolls is stable in this way and the other unstable.

The study is motivated by a desire to understand better the dynamical means by which a large mass of conducting fluid can create its own magnetism. It is argued that these instabilities suggest the existence of a mechanism of self-adjustment preventing λ from either increasing or decreasing indefinitely and noted that, very roughly, λ is of order unity in the earth's core.

1. Introduction

The work described below develops further an earlier study (Roberts & Stewartson 1974, which will be referred to here as RS) and is part of a continuing programme of research motivated by planetary and stellar magnetism. It has the twin objectives of understanding better the dynamical means by which large bodies of conducting fluid, such as the earth's core or the solar convection zone, can create their own magnetism, and of deciding what mechanical balance is struck that determines the average strength of their fields. A detailed introduction to the cosmical application may be found in RS, and § 6 below contains additional points of particular relevance to this paper.

Our model consists of a plane horizontal layer of inviscid fluid which is finitely conducting (in both the thermal and electrical senses) and is permeated by a horizontal field that corotates with the layer about the vertical. The upper and lower surfaces are maintained at different constant temperatures, the upper surface being the cooler. The layer is initially in a state of relative rest, and the temperature contrast is slowly increased. When it reaches a certain critical value, the layer becomes marginally unstable to convective overturning. The type of motions that ensues depends in a complicated way on q, the ratio of the thermal and magnetic diffusivities, and λ, which represents the relative strength of the Coriolis and magnetic forces.

In this paper, we concentrate on cases in which $q < 2$ and $\lambda < 2/3^{\frac{1}{2}}$, where q and λ are defined by equation (2.3) below. When $q < 2$, the principle of the exchange of stabilities holds, i.e. the convection is steady. When $\lambda < 2/3^{\frac{1}{2}}$ a degeneracy arises in the linear stability problem which is absent when $\lambda > 2/3^{\frac{1}{2}}$: this consists of two distinct eigenfunctions corresponding to two rolls inclined to the direction of the applied field at equal but opposite angles.

On increasing the temperature gradient slightly above the critical value, weakly nonlinear convection occurs. This was studied in RS, and a fundamental difficulty was encountered. It was found that, if the initial perturbation contained components of both the critical roll solutions described above, their nonlinear interaction created a force that could not be balanced by the pressure gradient, Coriolis forces or Lorentz forces, i.e. the governing equations became inadequate for the study. The inertial terms, which were entirely omitted from RS, are therefore restored in the present paper, and the double-roll solutions are considered *de novae.*

We find that a double-roll solution generally implies the existence of a geostrophic motion, parallel to the field and independent of the height; we call this 'the Taylor shear' in recognition of a pioneering paper by Taylor (1963) in which the significance of the flow was first realized. Taylor's prescription for following the evolution of the shear is to add that shear whose associated magnetic effect, at the instant under consideration, exactly cancels out the unbalanced forces from other components of the flow, such as those noted above that arise from roll interaction. It rests on the idea that the (magnetically modified) inertial and Alfvén waves have a very short time scale compared with other processes and evolve so quickly that they can continually preserve the balance of forces just mentioned. The reader is referred to Taylor's paper for a full description of the method.

We do not follow this prescription. We explicitly retain the inertial terms and so automatically all relevant wave motions. Although this has, compared with Taylor's procedure, the disadvantage of a greater analytic complexity, it has the advantage of leaving open the possibility that the short-period waves are unstable, and that the process of continuous adjustment visualized by Taylor may not in fact take place. Interestingly enough, we find some cases in which Taylor's procedure would omit results that appear to us to be possibly of geophysical significance. This topic is raised again in §6.

The plan of this paper is the following. In §2 the equations are written in a dimensionless form like that used in RS, but the geostrophic flow and inertial terms are included. The marginal stability problem of RS is reviewed in §3, and the manner in which the convective double roll gives rise to a Taylor shear is described. Two questions are asked in §4. If $\lambda < 2/3^{\frac{1}{2}}$ and only a single oblique roll is present, will a perturbation in the form of the other (missing) oblique roll grow or decay? And what (if instability occurs) will the character and the growth rate of the instability be? Related questions are posed in §5. In what way is the degeneracy of the linear stability problem considered by RS lifted by the presence of a small geostrophic shear? What effect does such a shear have on the critical temperature gradient for convection?

2. The governing equations

As in RS we are concerned with the stability of an inviscid fluid confined between rigid horizontal plates maintained at different temperatures and with the hotter plate beneath the cooler. The whole system rotates about the vertical axis and a uniform magnetic field acts in a fixed horizontal direction relative to the rotating frame. Define an orthogonal set of axes $Ox^*y^*z^*$ relative to the rotating frame, with Oz^* downwards, Oy^* in the direction of the applied field and O in the mid-plane. Let the fluid velocity be $\mathbf{V}^*$, the magnetic field $\mathbf{B}^*$ and the temperature θ^*. Introduce dimensionless variables by the transformations

$$\left.\begin{aligned} t^* = \frac{d^2}{\kappa\pi}t, \quad \mathbf{r}^* = (x^*,y^*,z^*) = \frac{d}{\pi}(x,y,z) = \frac{d}{\pi}\mathbf{r},\\ \mathbf{V}^* = \frac{\pi\kappa}{d}\epsilon\mathbf{V}(\mathbf{r},t), \quad \mathbf{B}^* = B_0[\hat{\mathbf{y}} + \mu\sigma\kappa\epsilon\mathbf{B}(\mathbf{r},t)],\\ \theta^* = \theta_0 + (\beta d/\pi)[z+\epsilon\theta(\mathbf{r},t)]. \end{aligned}\right\} \tag{2.1}$$

Here d is the distance between the plates, σ is the electrical conductivity of the fluid, μ the permeability, κ the thermal diffusivity, B_0 the magnitude of the imposed magnetic field, θ_0 the undisturbed temperature at the origin, β the undisturbed temperature gradient between the plates, $\hat{\mathbf{y}}$ a unit vector in the direction of y increasing and ϵ ($\ll 1$) a parameter representing the magnitude of the disturbance applied at $t = 0$.

The equations governing the motion of the fluid now assume the form

$$\operatorname{div}\mathbf{V} = \operatorname{div}\mathbf{B} = 0, \tag{2.2a}$$

$$\nabla^2\theta - V_z - \partial\theta/\partial t = \epsilon(\mathbf{V}.\nabla)\theta, \tag{2.2b}$$

$$\partial\mathbf{V}/\partial y + \nabla^2\mathbf{B} - q\,\partial\mathbf{B}/\partial t = -\epsilon q\operatorname{curl}(\mathbf{V}\times\mathbf{B}), \tag{2.2c}$$

$$\partial\mathbf{B}/\partial y - \lambda\hat{\mathbf{z}}\times\mathbf{V} - \lambda R\theta\hat{\mathbf{z}} - \operatorname{grad}\Pi = -\epsilon q\operatorname{curl}\mathbf{B}\times\mathbf{B} + \delta^2\partial\mathbf{V}/\partial t + \epsilon\delta^2(\mathbf{V}.\nabla)\mathbf{V}. \tag{2.2d}$$

Here Π is a dimensionless reduced pressure and

$$\lambda = \frac{2\Omega\rho_0}{\sigma B_0^2}, \quad R = \frac{g\alpha\beta d^2}{2\kappa\Omega\pi^2}, \quad q = \mu\sigma\kappa, \quad \delta^2 = \frac{\lambda\kappa\pi^2}{2\Omega d^2}, \tag{2.3}$$

where Ω is the angular velocity of the fluid, ρ_0 the density, α the coefficient of volume expansion, g the acceleration due to gravity and $\hat{\mathbf{z}}$ a unit vector in the downward vertical direction. Following RS we assume that $\alpha\beta d$, $\delta \ll 1$. In addition we shall, at an appropriate place, assume that $\delta = O(\epsilon q)$. The neglect of viscous effects implies that the Ekman number $\nu\pi^2/\Omega d^2$ is vanishingly small, ν being the kinematic viscosity. The boundary conditions associated with these equations are that

$$V_z = \theta = B_z = \partial B_x/\partial z = \partial B_y/\partial z = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}\pi, \tag{2.4}$$

since the plates are fixed and are perfect conductors, both electrically and thermally. In addition $\mathbf{V}$, $\mathbf{B}$ and θ are assumed to be periodic in x and y for modal disturbances and to remain bounded as $x^2+y^2 \to \infty$ for centred disturbances.

Once δ is neglected in (2.2d) it follows immediately that $T = 0$, where

$$\frac{1}{d} B_0^2 T \equiv \hat{\mathbf{z}} . \int_{-\frac{1}{2}d}^{\frac{1}{2}d} dz^* \operatorname{curl}^* (\operatorname{curl}^* \mathbf{B}^* \times \mathbf{B}^*), \tag{2.5}$$

and this was referred to in RS as Taylor's condition (Taylor 1963). It was noted that, when $\delta = 0$, (2.2) and (2.4) cannot determine the evolution of arbitrarily assigned $\mathbf{V}$, $\mathbf{B}$ and θ but only those for which $T = 0$. Attention was focused on marginally unstable solutions of (2.2) for which, for some reason, $T = 0$. In this paper we shall examine a class of motions in which the heuristic approach leads to a non-zero T, and discuss how this contradiction might be overcome.

It may be observed that the physical time scale represented by t is geophysically long: taking d to be the core radius, t^* is 10^{10} yr when $t = 1$. It should not be forgotten, however, that q is extremely small in the core ($q \doteq 3 \times 10^{-6}$) and that a theory that properly takes this into account may show that $t = q$ is the relevant time scale, giving a t^* of only 37 000 yr. The magnitude of the velocity perturbation in (2.1) is $O(\epsilon\delta^2/\lambda)$ relative to the solid-body rotation; it is much smaller than $O(\epsilon q)$, the size of the magnetic field perturbations in comparison with B_0. An initial $O(\epsilon)$ perturbation of $\mathbf{B}_0$ would, in the absence of Coriolis forces, generate Alfvén waves in a time of order $q^{\frac{1}{2}}\delta$, and fluid motions of order $\epsilon q^{\frac{1}{2}}/\delta$ would result. For the earth's core this time scale is a few years. It is assumed, so far without proof, that this perturbation decays rapidly in terms of t, especially in $\mathbf{V}$, so that, when $t = O(1)$, $\mathbf{V} = O(1)$. The fall in the value of $|\mathbf{V}|$ means that new terms of the governing equation eventually become significant and when that happens there is a possibility of an instability setting in of the type discussed by Eltayeb (1972) and RS, which leads to a resurgence of the perturbation in a time $O(1)$.

Alfvén waves of the type just described appear to be drastically affected by Coriolis forces. We find, however, that the perturbation field $q\epsilon B_0\mathbf{B}$ in (2.1) contains components that do travel with the phase velocity of Alfvén waves, appropriately reduced by the factor $q\epsilon$, i.e. they are associated with the time scale

$q^{\frac{1}{2}}\delta/\epsilon q$; see equation (4.31) below. We shall call these 'pseudo-Alfvén waves'. They appear to be associated with the torsional geostrophic waves which Braginskiĭ (1970) has postulated exist in the earth's core.

If in the governing equations (2.2) we set $\epsilon = \delta = 0$ it may easily be seen that there is a simple solution in which $\mathbf{B} = \theta = 0$ and $\mathbf{V} = F(x,t)\hat{\mathbf{y}}$. Thus there are two classes of solution to these reduced equations, one of Eltayeb's type and one consisting of a motion down the lines of force; not only could the second class be a residue of the larger velocities that must exist when $t = O(\delta)$ but there is no *a priori* reason why it should ever be small. For the class of disturbances considered in RS, $T = 0$ and there is no need to insist that $F \neq 0$. For the class to be considered here however $T \neq 0$, and it turns out that $F = O(\epsilon^{-1})$.

3. Marginal instability due to a pair of oblique rolls

It was shown by Eltayeb (1972) and RS (§3) that if $\lambda < 2/3^{\frac{1}{2}}$ and $q < 2$ the solution $\mathbf{V} = \mathbf{B} = \theta = 0$ of (2.2) is in neutral equilibrium if $R = 3^{\frac{3}{2}}$ provided that we set $\delta = 0$. The critical solution of the linearized equations obtained by also omitting all terms $O(\epsilon)$ from (2.2) is given by

$$\left.\begin{aligned}\mathbf{V}_1 &= [A_1(l+3^{\frac{1}{2}}m)\sin\phi_1\sin z,\quad A_1(m-3^{\frac{1}{2}}l)\sin\phi_1\sin z,\quad 2A_1\cos\phi_1\cos z],\\ \mathbf{B}_1 &= \tfrac{1}{3}m[A_1(l+3^{\frac{1}{2}}m)\cos\phi_1\sin z,\quad A_1(m-3^{\frac{1}{2}}l)\cos\phi_1\sin z,\quad -2A_1\sin\phi_1\cos z],\\ \theta_1 &= -\tfrac{2}{3}A_1\cos\phi_1\cos z.\end{aligned}\right\}\tag{3.1}$$

Here $$\phi_1 = lx+my+\alpha_1,$$

l and m are constants satisfying $m^2 = 3^{\frac{1}{2}}\lambda$ and $l^2+m^2 = 2$, and A_1 and α_1 are arbitrary constants. The notation is slightly different from that of equation (3.8) of RS but it is hoped that no confusion will arise. Without loss of generality we may take $l > 0$ in (3.1) and construct another solution of the linearized equations by changing the sign of m, namely

$$\left.\begin{aligned}\mathbf{V}_2 &= [A_2(l-3^{\frac{1}{2}}m)\sin\phi_2\sin z,\quad -A_2(m+3^{\frac{1}{2}}l)\sin\phi_2\sin z,\quad 2A_2\cos\phi_2\cos z],\\ \mathbf{B}_2 &= \tfrac{1}{3}m[-A_2(l-3^{\frac{1}{2}}m)\cos\phi_2\sin z,\quad A_2(m+3^{\frac{1}{2}}l)\cos\phi_2\sin z,\quad 2A_2\sin\phi_2\cos z],\\ \theta_2 &= -\tfrac{2}{3}A_2\cos\phi_2\cos z,\end{aligned}\right\}\tag{3.2}$$

where $$\phi_2 = lx-my+\alpha_2,$$

and A_2 and α_2 are constants.

When $R = 3^{\frac{3}{2}}$ the static state is stable to all other small disturbances, and when $R < 3^{\frac{3}{2}}$ it is stable to all disturbances. In a marginal state of instability, when $R-3^{\frac{3}{2}}$ is small and positive, the most unstable modes are also given by (3.1) and (3.2) although $\log A_1$ and $\log A_2$ are then linearly increasing functions of $(R-3^{\frac{3}{2}})\,t$. These solutions take the form of simple rolls inclined to the direction of the undisturbed magnetic field and the nonlinear evolution of one of them, uninfluenced by the others, is discussed in RS. Clearly such an evolutionary study is of restricted value because, from a physical standpoint, arbitrary initial disturbances are permissible, and it would be exceptional if A_1A_2 turned out to be zero. Once this restriction is removed, however, the mathematical problem becomes much more difficult and the reason is that Taylor's condition is violated.

When $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$, we have

$$\begin{aligned}
&q\epsilon \operatorname{curl} \mathbf{B} \times \mathbf{B} = \\
&\tfrac{1}{3}\epsilon q\lambda A_1^2[-(m+3^{\frac{1}{2}}l\cos 2z)\sin 2\phi_1,\ (l-3^{\frac{1}{2}}m\cos 2z)\sin 2\phi_1,\ -3^{\frac{1}{2}}\sin 2z(1+\cos 2\phi_1)] \\
&+\tfrac{1}{3}\epsilon q\lambda A_2^2[(m-3^{\frac{1}{2}}l\cos 2z)\sin 2\phi_2,\ (l+3^{\frac{1}{2}}m\cos 2z)\sin 2\phi_2,\ -3^{\frac{1}{2}}\sin 2z(1+\cos 2\phi_2)] \\
&+\tfrac{2}{3}\epsilon q\lambda A_1A_2[-m\sin(\phi_1-\phi_2)+3^{\frac{1}{2}}l\sin(\phi_1+\phi_2)\cos 2z,\ -l\sin(\phi_1+\phi_2) \\
&\qquad -3^{\frac{1}{2}}m\sin(\phi_1-\phi_2)\cos 2z,\ (l^2-m^2)\,3^{\frac{1}{2}}\cos\phi_1\cos\phi_2\cos 2z]. \qquad (3.3)
\end{aligned}$$

On regarding (3.3) as a forcing term in (2.4) which helps to provide $O(\epsilon)$ contributions to $\mathbf{V}$, $\mathbf{B}$ and θ, and keeping $\delta = 0$, we see that, except for the part

$$-\tfrac{2}{3}\epsilon q\lambda l A_1A_2[0, \sin(\phi_1+\phi_2), 0] = -\tfrac{2}{3}\epsilon q\lambda l A_1A_2[0, \sin(2lx+\alpha_1+\alpha_2), 0], \qquad (3.4)$$

every term in (3.3) can be accounted for by assuming that its contribution to the final solution has the dependence on x, y and z indicated. In order to account for (3.4), however, we would need a contribution independent of y and z and in particular a contribution to $\mathbf{V}$ of

$$-\tfrac{2}{3}\epsilon q l A_1A_2[\sin(2lx+\alpha_1+\alpha_2), 0, 0], \qquad (3.5)$$

which violates the equation of continuity. Further, (3.4) violates the Taylor condition while the other parts of (3.3) are compatible with it.

If one or both of $q < 2$ and $\lambda < 2/3^{\frac{1}{2}}$ is not satisfied this difficulty does not seem to occur with the critical modes or at least it is not so serious. Thus, if $3^{\frac{1}{2}}\lambda > 2$ and $q < 2$, or if $3^{\frac{1}{2}}\lambda(1+q) > 2$ and $q > 2$, the marginally unstable modes have $l = 0$, and so the critical component of curl $\mathbf{B}\times\mathbf{B}$ vanishes. On the other hand if $3^{\frac{1}{2}}\lambda(1+q) < 2$ and $q > 2$ we again have marginally unstable oblique rolls at slightly supercritical values of R but now α_1 and α_2 are linear functions of time. Although in the case of modal disturbances like (3.1) and (3.2) we may expect a term like (3.4) leading to a similar difficulty, it appears that centred disturbances give rise to modulated forms of (3.1) and (3.2) in which A_1 and A_2 are slowly varying functions of x, y and t at large values of t. Each of these would move with the appropriate group velocity in a different direction, so that by the time they were dominant they would be non-zero only in non-overlapping domains.

We claim that the compatibility of (3.3) with Taylor's condition may be restored by introducing a velocity component

$$[0, (\epsilon q)^{-1}F_\infty(x), 0] \qquad (3.6)$$

into the undisturbed state of the fluid, where F_∞ is hopefully determinate in terms of the initial disturbance. In physical terms this corresponds to a drift of velocity $\Omega d(2\lambda\pi q)^{-1}F_\infty\delta^2$, along the lines of force. It should be noticed that this velocity is independent of ϵ, although it still remains small in comparison with the basic rotational velocity, a characteristic value for which is Ωd, and in order to derive an expression for F_∞ in particular cases we have to assume that $\delta \ll \epsilon$. We shall refer to (3.6) as the *Taylor shear*. His basic idea was that, if (2.5) is violated as the initial disturbance to the fluid develops, then there is a back reaction on the basic flow which sets up a Taylor shear which compels the disturbance to become compatible with (2.5).

4. The induced Taylor shear

As explained in §2 the first stage in the evolution of an initial disturbance is the setting up of Alfvén waves for which $\mathbf{V} = O(\delta^{-1}\mathbf{B})$, the time taken being $O(\delta)$. It is *hoped* that these decay while t is still small until $\mathbf{V} = O(\mathbf{B})$, and that thereafter δ may be neglected in (2.2d). If, however, $T \neq 0$ as the initial stages of the evolution of the disturbances come to an end, we claim that the fluid acquires a relatively large velocity $F(x,t)/\epsilon q$ parallel to the lines of force, the Taylor shear, through the term $\delta^2 \partial V_y/\partial t$ in (2.2d) and it is desirable that F builds up to a steady value such that the corresponding modifications to the flow make $T = 0$. If we write

$$p = \delta/\epsilon q, \tag{4.1}$$

then when $p = O(1)$ the build-up of F takes place in a time $O(1)$. For F makes an $O(F)$ contribution to $\mathbf{B}$ from (2.2c) and hence a contribution to $\epsilon \operatorname{curl} \mathbf{B} \times \mathbf{B}$ which is $O(\epsilon q F)$. Thus we have to balance an $O(\epsilon q)$ term from the original non-zero value of T, one which is $O(\epsilon q F)$ from the modification due to F, and an $O(\delta^2 \dot{F}/\epsilon q)$ term from the right-hand side of (2.2d). It follows that if F tends to a limit it must be $O(1)$ as anticipated by (3.6), and when $p = O(1)$ the time scale on which the limit is achieved is also $O(1)$ using (4.1). There is however no *a priori* reason to guarantee that F remains well behaved and in the example we shall study below this will be seen not to be invariably the case.

Although any pair of matched oblique rolls will produce a non-zero value of T greatest interest centres round a marginally unstable pair at $R = 3^{\frac{3}{2}}+$ since in the absence of a Taylor shear or if the Taylor shear takes some time to develop the natural evolution of any weak disturbance will lead to the emergence of such pairs as the dominant form of the disturbance. Further, in order to pose an analytically tractable problem we shall suppose that one of the pairs is much weaker than the other. Thus let us suppose that at time $t = 0$ the fluid is given a disturbance defined by (3.1) and (3.2) with $A_2 = \Delta A_1$ and $|\Delta| \ll 1$. The Taylor shear needed to cancel (3.4) then makes a weak contribution to $\mathbf{B}$ and its effect may be regarded as a small perturbation on (3.1). We note in parentheses that other pairs may be considered (i.e. not obeying $m^2 = 3^{\frac{1}{2}}\lambda$ and $l^2 + m^2 = 2$) using parallel arguments. We write

$$\mathbf{V} = \mathbf{V}_1 + \Delta\mathbf{U} + \Delta\tilde{\mathbf{V}} + O(\epsilon), \tag{4.2a}$$

$$\mathbf{B} = \mathbf{B}_1 + \Delta\tilde{\mathbf{B}} + O(\epsilon), \tag{4.2b}$$

$$\theta = \theta_1 + \Delta\tilde{\theta} + O(\epsilon), \tag{4.2c}$$

$$\mathbf{U} = [0, (\epsilon q)^{-1} F(x,t), 0]. \tag{4.2d}$$

Here $\tilde{\mathbf{V}}$, $\tilde{\mathbf{B}}$ and $\tilde{\theta}$ are functions of $\mathbf{r}$ and t to be found, and we assume that when

$t = 0$ they are given by (3.2) with A_2 replaced by A_1. In addition and without any loss of generality we take $\alpha_1 = 0$ and $\alpha_2 = \frac{1}{2}\pi$. The Taylor shear $\mathbf{U}$ is a function of x and t only and $F(x, 0) \equiv 0$. On substituting (4.2) into (2.2) we obtain the following set of equations:

$$\operatorname{div}\tilde{\mathbf{V}} = \operatorname{div}\tilde{\mathbf{B}} = 0, \tag{4.3a}$$

$$\nabla^2\tilde{\theta} - \tilde{V}_z - \partial\tilde{\theta}/\partial t = q^{-1}F\,\partial\theta_1/\partial y, \tag{4.3b}$$

$$\partial\tilde{\mathbf{V}}/\partial y + \nabla^2\tilde{\mathbf{B}} - q\,\partial\tilde{\mathbf{B}}/\partial t = F\,\partial\mathbf{B}_1/\partial y - (B_{1x}\partial F/\partial x)\,\hat{\mathbf{y}}, \tag{4.3c}$$

$$\partial\tilde{\mathbf{B}}/\partial y - \lambda\hat{\mathbf{z}}\times\tilde{\mathbf{V}} - \lambda R\tilde{\theta}\hat{\mathbf{z}} - \operatorname{grad}\tilde{\Pi} = 0, \tag{4.3d}$$

with $\lambda < 2/3^{\frac{1}{4}}$, $R = 3^{\frac{3}{2}}$ and the dependent variables satisfying (2.4) as is invariably the case here. With any choice of F, particularly that of equation (4.5) below, these equations may now be solved completely and used to compute $\epsilon q(\operatorname{curl}\mathbf{B})\times\mathbf{B}$. As explained in §3 all components of this vector may be cancelled by $O(\epsilon)$ contributions to $\mathbf{V}$, $\mathbf{B}$ and θ except that part which gives rise to a non-zero value of T. The function $\mathbf{U}$ makes two contributions to (2.2d), one through $\hat{\mathbf{z}}\times\mathbf{V}$, which may be cancelled by adding a suitable function of x and t to Π, and one through $\delta^2\partial\mathbf{V}/\partial t$, which must be cancelled by T. Thus

$$(\delta/q\epsilon)^2\,\partial F/\partial t = \langle(\operatorname{curl}\mathbf{B}_1)\times\tilde{\mathbf{B}} + (\operatorname{curl}\tilde{\mathbf{B}})\times\mathbf{B}_1\rangle_y, \tag{4.4}$$

where $\langle\mathbf{G}\rangle_y$ denotes the mean value, with respect to y and z, of the y component of $\mathbf{G}$. The particular problem posed in this section can be solved by writing

$$F = 2f(t)\cos 2lx, \tag{4.5}$$

where f is a function of t only and $f(0) = 0$.

It is convenient to write

$$\mathbf{V}_1 = (iU_1\sin z, iV_1\sin z, W_1\cos z)\,e^{i(lx+my)} + \text{c.c.}, \tag{4.6}$$

where c.c. denotes the complex conjugate and U_1, V_1 and W_1 are constants. In fact from (3.1) we have

$$U_1 = -\tfrac{1}{2}A_1(l + 3^{\frac{1}{2}}m), \quad V_1 = -\tfrac{1}{2}A_1(m - 3^{\frac{1}{2}}l), \quad W_1 = A_1. \tag{4.7}$$

Continuing we write

$$\begin{aligned}\tilde{\mathbf{V}} = \{&[3^{\frac{1}{2}}q^{-1}m^2A_1(U_2e^{-ilx} + U_3e^{3ilx}) + \tfrac{1}{2}A_1(l - 3^{\frac{1}{2}}m)\,e^{-ilx}]\,e^{imy}\sin z,\\ &[3^{\frac{1}{2}}q^{-1}m^2A_1(V_2e^{-ilx} + V_3e^{3ilx}) - \tfrac{1}{2}A_1(m + 3^{\frac{1}{2}}l)\,e^{-ilx}]\,e^{imy}\sin z,\\ &[-3^{\frac{1}{2}}iq^{-1}m^2A_1(W_2e^{-ilx} + W_3e^{3ilx}) - iA_1e^{-ilx}]\,e^{imy}\cos z\} + \text{c.c.},\end{aligned} \tag{4.8}$$

where U_2, U_3, V_2, V_3, W_2 and W_3 are functions of t only, all of which vanish at $t = 0$. In a similar way we define

$$\mathbf{B}_1 = (X_1\sin z, Y_1\sin z, -iZ_1\cos z)\,e^{i(lx+my)} + \text{c.c.}, \tag{4.9}$$

X_1, Y_1 and Z_1 being constants whose values follow at once from (3.1); a parallel expression to (4.8) for $\tilde{\mathbf{B}}$; and

$$\begin{aligned}\theta = \{&\theta_1\cos z\,e^{i(lx+my)} + [-3^{\frac{1}{2}}iq^{-1}m^2A_1(\theta_2e^{-ilx} + \theta_3e^{3ilx}) + \tfrac{1}{3}iA_1e^{-ilx}]\,e^{imy}\cos z\\ &+ \text{c.c.} + O(\epsilon)\},\end{aligned} \tag{4.10}$$

where θ_1 $(= -\frac{1}{3}A_1)$ is now a constant, and θ_2 and θ_3 are functions of t that are initially zero.

Substitute these formulae into (4.3) and take Laplace transforms with respect to t. Distinguish Laplace transforms by an overbar, and let s be the parameter of the transformation. Then two sets of equations are obtained. The one governing $\bar{U}_2$, $\bar{V}_2$, $\bar{W}_2$, $\bar{X}_2$, $\bar{Y}_2$, $\bar{Z}_2$ and $\bar{\theta}_2$ simplifies to

$$\left.\begin{aligned}(3+s)\,\bar{\theta}_2+\bar{W}_2 &= -\bar{f}/3^{\frac{3}{2}}m,\\ (3+qs)\,\bar{X}_2+m\bar{U}_2 &= (l+3^{\frac{1}{2}}m)\,q\bar{f}/6\times 3^{\frac{1}{2}},\\ (3+qs)\,\bar{Z}_2+m\bar{W}_2 &= -q\bar{f}/3^{\frac{3}{2}},\\ 3^{\frac{3}{2}}\bar{X}_2 &= -l\bar{U}_2+(1+m^2)\,\bar{W}_2+3^{\frac{3}{2}}lm\bar{\theta}_2,\\ 3^{\frac{3}{2}}\bar{Z}_2 &= -2\bar{U}_2+l\bar{W}_2+6\times 3^{\frac{1}{2}}m\bar{\theta}_2,\end{aligned}\right\}\tag{4.11}$$

the equation for $\bar{V}_2$ and $\bar{Y}_2$ following from the equation of continuity (4.3*a*). The set for $\bar{U}_3$, $\bar{V}_3$, $\bar{W}_3$, $\bar{X}_3$, $\bar{Y}_3$, $\bar{Z}_3$ and $\bar{\theta}_3$ reduces to

$$\left.\begin{aligned}(3+8l^2+s)\,\bar{\theta}_3+\bar{W}_3 &= -\bar{f}/3^{\frac{3}{2}}m,\\ (3+8l^2+qs)\,\bar{X}_3+m\bar{U}_3 &= (l+3^{\frac{1}{2}}m)\,q\bar{f}/6\times 3^{\frac{1}{2}},\\ (3+8l^2+qs)\,\bar{Z}_3+m\bar{W}_3 &= -q\bar{f}/3^{\frac{3}{2}},\\ 3^{\frac{1}{2}}(3+8l^2)\,\bar{X}_3 &= 3l\bar{U}_3+(1+m^2)\,\bar{W}_3-3^{\frac{3}{2}}lm\bar{\theta}_3,\\ 3^{\frac{1}{2}}(3+8l^2)\,\bar{Z}_3 &= -(2+8l^2)\,\bar{U}_3-3l\bar{W}_3+3^{\frac{3}{2}}m(2+8l^2)\,\bar{\theta}_3.\end{aligned}\right\}\tag{4.12}$$

It may easily be shown that, if

$$\mathbf{B} = (X\sin z,\ Y\sin z,\ -iZ\cos z)\,e^{imy}+\text{c.c.},\tag{4.13}$$

where X, Y and Z are complex functions of x and t only, then

$$\begin{aligned}T &= \langle \operatorname{curl}\mathbf{B}\times\mathbf{B}\rangle_y\\ &= \frac{1}{2m}\frac{\partial}{\partial x}\left\{i\left(X_{\text{c.c.}}\frac{\partial X}{\partial x}-X\frac{\partial X_{\text{c.c.}}}{\partial x}\right)-(XZ_{\text{c.c.}}+X_{\text{c.c.}}Z)\right\},\end{aligned}\tag{4.14}$$

where $X_{\text{c.c.}}$ and $Z_{\text{c.c.}}$ are the complex conjugates of X and Z. Hence, bearing in mind that $|\Delta| \ll 1$, (4.4) reduces to

$$\begin{aligned}2s(\delta/q\epsilon)^2\bar{f}+(lm^2A_1^2/3^{\frac{1}{2}}q)\,\{[2\bar{X}_2-(l+3^{\frac{1}{2}}m)\,\bar{Z}_2]\\ -[2\bar{X}_3-(l+3^{\frac{1}{2}}m)\,\bar{Z}_3-4l(l+3^{\frac{1}{2}}m)\,\bar{X}_3]\} = -2lm^2A_1^2/3^{\frac{5}{2}}s.\end{aligned}\tag{4.15}$$

We notice that, if $\alpha_1 \neq 0$ or $\alpha_2 \neq \frac{1}{2}\pi$, an identical formula to (4.15) is obtained if we replace (4.5) by $F = 2f(t)\sin(2lx+\alpha_1+\alpha_2)$. It should be particularly observed that according to (4.4) and (4.14) no $4lx$ harmonics will be created in F, so that (4.5) is justified *a posteriori*.

To make further progress, it is necessary to solve (4.11) and (4.12) and substitute the resulting expressions into (4.15). After some algebraic reductions, we find that (4.11) gives

$$\begin{aligned}\bar{X}_2/\Delta_2\bar{f} = (1-q)\,[m(3+qs)-3^{\frac{1}{2}}l]+(q/6\times 3^{\frac{1}{2}}m)\,[m(l+3^{\frac{1}{2}}m)\,(3+s)\,(3+qs)\\ +6\times 3^{\frac{1}{2}}m^2(3+qs)-3^{\frac{1}{2}}(2l^2+3m^2+3^{\frac{1}{2}}lm)\,(3+s)],\end{aligned}\tag{4.16}$$

$$\bar{Z}_2/\Delta_2\bar{f} = -2\times 3^{\frac{1}{2}}(1-q)-(q/3^{\frac{3}{2}}m)\,(3+s)\,[m(6+qs)+2\times 3^{\frac{1}{2}}l],\tag{4.17}$$

where

$$1/\Delta_2 = s[18+3q(2+q)\,s+q^2s^2]. \tag{4.18}$$

The similar expressions for $\bar{X}_3$ and $\bar{Z}_3$ that follow from (4.12) will not be given explicitly. They are needed only to reduce (4.15) to the form

$$\begin{aligned}(\delta/\epsilon m A_1)^2 s\bar{f} + \llbracket \Delta_2[&m(1-q)\,(6+qs)+(q/3^{\frac{3}{2}})\,\{l(3+s)\,(6+qs)+3^{\frac{1}{2}}m(6+s)\,(3+qs)\}] \\ &-\Delta_3[m(1-q)\,\{(2+8l^2+qs)\,(1-2l^2-2\times 3^{\frac{1}{2}}lm)+4(1-2l^2)\} \\ &+(q/3^{\frac{3}{2}})\,\{(l+3^{\frac{1}{2}}m)\,(1-l^2-3^{\frac{1}{2}}lm)\,(3+8l^2+qs)\,(3+8l^2+s) \\ &+3^{\frac{3}{2}}m(1-2l^2-2\times 3^{\frac{1}{2}}lm)\,(3+8l^2+qs)+3l(1-4l^2)\,(3+8l^2+s)\}]\rrbracket\,(l^2\bar{f}/3^{\frac{1}{2}}) \\ =&-lq^2/3^{\frac{3}{2}}s, \end{aligned} \tag{4.19}$$

where

$$1/\Delta_3 = (3+8l^2+s)\,(3+8l^2+qs)^2-9(2+8l^2)\,(3+8l^2+qs)+3(3+8l^2)\,(3+8l^2+s). \tag{4.20}$$

Let us suppose there is a stable final state. It may be found by letting $s \to 0$ in (4.16)–(4.20). It then follows that

$$X_2(\infty) = \frac{q}{6\times 3^{\frac{1}{2}}m}(l-3^{\frac{1}{2}}m), \quad Z_2(\infty) = \frac{q}{3^{\frac{3}{2}}m}, \quad \bar{f}(0) = -\frac{3^{\frac{1}{2}}q}{ql+3^{\frac{1}{2}}m}. \tag{4.21}$$

Referring to (4.8), we see that, when the second roll is initially weak relative to the first, the Taylor shear evolves in a way which ultimately causes the second roll to disappear entirely, and with it the need for the Taylor shear, which is therefore reduced to zero. Although this conclusion is based on an analysis of the marginally unstable rolls it can, not unexpectedly, be shown to hold for any similar pair of rolls. It would be interesting to discover whether the same conclusion holds when Δ is not small, but this is a more difficult undertaking, since it is not then possible to assume a simple expression such as (4.5) for F. It is necessary to express F as a general Fourier series even in x and with period π/l.

Probably a more significant question to ask is whether the limit (4.21) is attainable from an initial state in which $f = 0$. For this to be possible it is necessary that the zeros of the coefficient of $\bar{f}$ in (4.19) should lie to the left of the imaginary s axis. It should be pointed out at once that, perhaps surprisingly, the cases of positive and negative m are quite different. It is possible to find values of λ and q for which the real parts of the roots are all negative for $m > 0$, while one or more are positive for $m < 0$. Then, starting with a strong roll (A_1) with $m > 0$, and perturbing it with a weak roll ($A_2 = \Delta A_1$) for which $m < 0$, a transient Taylor shear is excited that obliterates the weak roll in an $O(1)$ time; but, starting with the $m < 0$ roll dominant, a perturbation in the form of the $m > 0$ roll grows on the $O(1)$ time scale until the condition $|\Delta| \ll 1$ is violated, and our analysis ceases to apply.

The general case involves locating the roots of a seventh-order polynomial equation having complicated coefficients. We shall first illustrate the situation by considering two special cases, $q \to 0$ and $q = 1$, in which the polynomial reduces, respectively, to a cubic and a quartic.

Taking $p^2q = O(1)$, we find that as $q \to O$ the coefficient of $\bar{f}$ in (4.19) is proportional to

$$C_0(s) \equiv [s(4l^2+3)+32l^4]\left[\left(\frac{\delta}{q\epsilon A_1}\right)^2 s^2+\frac{lm^3}{3^{\frac{3}{2}}q}\right] + \frac{lm^3\,s}{3^{\frac{1}{2}}q(8l^2+3)}[(2l^2-1)(4l^2+3)+2\times 3^{\frac{1}{2}}lm(4l^2+1)]. \quad (4.22)$$

It is clear from (4.22) that, throughout the range $0 < \lambda < 2/3^{\frac{1}{2}}$ of interest, the roll with negative m is unstable. Also, even if $m > 0$, $C_0(s)$ possesses a root in the positive half-plane when

$$(2l^2-1)(4l^2+3)+2\times 3^{\frac{1}{2}}lm(4l^2+1) < 0, \quad (4.23)$$

i.e. when approximately

$$l < 0{\cdot}36115, \quad \text{i.e. } 1{\cdot}07940 < \lambda < 2/3^{\frac{1}{2}}. \quad (4.24)$$

This holds for finite values of p^2q. It may be shown however (see below) that, provided that $\delta/\epsilon q$ is sufficiently small, two of the four roots lost by assuming that $p^2q = O(1)$ have positive real parts for values of λ slightly smaller than those given in (4.24).

When $q = 1$, the coefficient of $\bar{f}$ in (4.19) is proportional to

$$C_1(s) \equiv [s^2+2(3+8l^2)\,s+64l^4]\left[\left(\frac{\delta}{q\epsilon A_1}\right)^2 s^2+\frac{lm^2(l+3^{\frac{1}{2}}m)}{9}\right] - \frac{lm^2(l+3^{\frac{1}{2}}m)}{9}[1-l(l+3^{\frac{1}{2}}m)]\,s(s+6+8l^2). \quad (4.25)$$

It is clear from (4.25) that rolls for which

$$l+3^{\frac{1}{2}}m < 0 \quad (4.26)$$

are unstable and that, even if $l+3^{\frac{1}{2}}m > 0$, $C_1(s)$ possesses a root in the positive half-plane when

$$l(l+3^{\frac{1}{2}}m) < 1. \quad (4.27)$$

In other words, the 'negative-m roll' (the roll for which $m < 0$) is unstable when

$$l < (3/2)^{\frac{1}{2}}, \quad \text{i.e. } 3^{\frac{1}{2}}/6 < \lambda < 2/3^{\frac{1}{2}}, \quad (4.28)$$

and both rolls are unstable when

$$l < \tfrac{1}{2}(3^{\frac{1}{2}}-1), \quad \text{i.e. } \lambda_1 \equiv 2^{-1}+3^{-\frac{1}{2}} < \lambda < 2/3^{\frac{1}{2}}. \quad (4.29)$$

It should be noted that all the criteria we have obtained for $q \to 0$ and $q = 1$ have been independent of p (i.e. of ϵA_1 and δ).

It is easy to see from the sign of the constant coefficient of the septic equation that arises for general q that the negative-m roll is unstable for

$$ql+3^{\frac{1}{2}}m < 0, \quad \text{i.e. } \lambda > \lambda_0 \equiv 2q^2/3^{\frac{1}{2}}(q^2+3). \quad (4.30)$$

By considering, in the particular case of small p, the coefficients of s^7, s^6, s^5 and s^4 in the governing septic equation obtained from (4.19), we find that two of its roots are, to leading order,

$$s = \pm\, i(qm\epsilon A_1/3\delta)^{\frac{1}{2}}\,q^{-\frac{1}{2}}l(l+3^{\frac{1}{2}}m)+s_1, \quad (4.31)$$

where s_1 is of order unity and changes sign from negative to positive as λ increases through λ_1. We may conclude that the 'positive-m roll' (the roll for which $m > 0$) is necessarily unstable at small p whenever $\lambda > \lambda_1$. That this is not necessarily true for all p may be appreciated from the case $q = 0$, in which $\lambda_1 = 1{\cdot}0773$ although (because of the existence of another pair of roots) the roll becomes unstable at large p when λ exceeds $1{\cdot}0794$, approximately. For $q = 1$, the instability occurs at $\lambda = \lambda_1$ for all p. It is conceivable (particularly for $q > 1$) that instability generally arises at some λ less than λ_1, but in § 5 we shall, for simplicity of presentation, regard $\lambda = \lambda_1$ as marking marginal stability for the positive-m roll for all p.

It should be clear from this discussion that the limit $q \to 0$ is not straightforward, but it is of relevance to core motions in the earth. We hope to discuss this limit further in a subsequent paper.

5. The effect of a weak Taylor shear on marginal instability

The difficulties that result from the introduction of a weak Taylor shear, namely the instabilities occurring in certain ranges of values of l and m, or equivalently of λ, are serious within their own context, but need not be disastrous to the whole theory of magneto-fluid oscillations on the time scale envisaged in § 2. For the shear is weak and it may be that as it grows in strength a feedback mechanism is set up which controls the instability and readjusts the flow field.† Further, the mechanism producing the shear, a weak double roll, is rather special and as soon as a more complicated perturbed field than (4.2) is considered the simplicity of (4.5) may be lost. The upshot might be for example that a neutrally stable state of the fluid can be obtained at some value of R but not necessarily $3^{\frac{3}{2}}$. In this section we shall initiate a study of how a Taylor shear can exert control over the critical modal disturbances.

Assume that the shear is defined by (4.2d) with F now a function of x only. The linearized equations governing small disturbances on the scale of § 2 are the same as (4.3) except that here we shall suppose that R is arbitrary. The possible choices of F depend of course on the evolution of the disturbance at small times when the scales of § 2 are inappropriate but we have in mind that they are such that $T = 0$ as $t \to 0$ on the scale of § 2. This is a slightly different situation from that of § 4 for there the initial *shear* was zero. Having found the modal solutions which satisfy $T = 0$ a possible second step might be to consider the stability of the solutions on time scales both longer and shorter than $t \sim 1$.

We write

$$\left.\begin{aligned} \mathbf{V} &= (iU\sin z, \quad iV\sin z, \quad W\cos z)\, e^{imy+i\omega t}, \\ \mathbf{B} &= (X\sin z, \quad Y\sin z, \quad -iZ\cos z)\, e^{imy+i\omega t}, \\ \theta &= \Theta\, e^{imy+i\omega t}, \end{aligned}\right\} \tag{5.1}$$

where U, V, W, X, Y, Z and Θ are now complex and functions of x alone. With

$$iD = d/dx, \tag{5.2}$$

† Roberts & Soward (1972, § 5) have shown that this happens in the case of MAC-waves.

we obtain from (4.3)

$$(D^2+m^2+1)\,\Theta+W = -i(\omega+q^{-1}mF)\,\Theta, \tag{5.3a}$$

$$(D^2+m^2+1)\,X+mU = -i(\omega+mF)\,X, \tag{5.3b}$$

$$(D^2+m^2+1)\,Z+mW = -i(\omega+mF)\,Z, \tag{5.3c}$$

$$DX+(m^2+1)\,Z-m\lambda R_c\Theta-\lambda U = m\lambda(R-R_c)\,\Theta, \tag{5.3d}$$

$$(D^2+m^2)\,X+DZ-\lambda W = 0, \tag{5.3e}$$

equations which are perhaps more fundamental than (4.11) and (4.12). A solution of (5.3) is required which is periodic in x and for which $T = 0$. For given F the periodicity condition fixes ω. The condition $T = 0$ can be simplified by using (4.14), which shows that, since there are generally two periodic solutions of (5.3) for given F, there is some freedom in the choice of F.

The programme is formidable and only a little progress towards realizing it can be reported here. We shall assume that

$$F = 2\Delta\cos 2lx, \tag{5.4}$$

where Δ is a small positive constant and $R-R_c = O(\Delta)$. We shall consider only the case ($l^2+m^2 = 2$, $m^2 = 3^{\frac{1}{2}}\lambda$, $R_c = 3^{\frac{3}{2}}$) in which the convection is marginal and steady ($\omega = 0$) according to the $\Delta = 0$ theory of RS. We then expand the unknowns in powers of Δ:

$$\left.\begin{aligned}\omega &= \Delta\omega_1+\Delta^2\omega_2+\dots,\\ W &= W^{(0)}(x)+\Delta W^{(1)}(x)+\Delta^2W^{(2)}(x)+\dots,\end{aligned}\right\} \tag{5.5}$$

with similar expressions for the other functions appearing in (5.3). The critical value $3^{\frac{3}{2}}$ of R for $\Delta = 0$ is now written as R_{c0}. The equations for the terms independent of Δ in these expansions coincide with (5.3) when the right-hand sides of (5.3) are set equal to zero. Hence we have

$$W^{(0)} = L_0e^{ilx}+M_0e^{-ilx}, \tag{5.6}$$

with corresponding forms for $U^{(0)}$, $X^{(0)}$, $Z^{(0)}$ and $\Theta^{(0)}$ that may be inferred from RS from (3.1) above. To avoid confusion in interpretation, we reiterate here that l is by convention necessarily positive. In § 4 the initial conditions placed the two possible rolls on a different footing, and it was necessary to allow m to have either sign. In this section, there is no such built-in symmetry and we may therefore assume without loss of generality that m is *negative*, this sign being chosen for ease of comparison with § 4. The coefficients A_1 and A_2 of that section, which correspond to the coefficients L_0 and M_0 in (5.6), represent respectively the negative-m roll and the positive-m roll.

The equations for the coefficients of Δ in (5.5) are exemplified by

$$(D^2+m^2+1)\,\Theta^{(1)}+W^{(1)} = -i[\omega_1+q^{-1}m(e^{2ilx}+e^{-2ilx})]\,\Theta^{(0)}, \tag{5.7}$$

which follows from (5.3a). It is clear that the solution for $W^{(1)}$ must be of the form

$$W^{(1)} = K_1e^{3ilx}+L_1e^{ilx}+M_1e^{-ilx}+N_1e^{-3ilx}. \tag{5.8}$$

Now l, m and R have been chosen such that the homogeneous equations for $W^{(0)}$, etc., have a non-trivial solution and so far L_0 and M_0 are arbitrary constants. As a result the linear algebraic equations for L_1 and M_1 have zero coefficients and the forcing terms from the right-hand side of (5.7) which are proportional to $e^{\pm ilx}$ must therefore vanish. For the marginally unstable case which we are studying here this is only possible if

$$\left.\begin{aligned} q[3^{\frac{1}{2}}\omega_1 + i\Delta^{-1}(R-R_{c0})]\,L_0 &= -(lq-3^{\frac{1}{2}}m)\,M_0,\\ q[3^{\frac{1}{2}}\omega_1 + i\Delta^{-1}(R-R_{c0})]\,M_0 &= (lq+3^{\frac{1}{2}}m)\,L_0. \end{aligned}\right\} \tag{5.9}$$

The equations for K_1 and L_1 are closely related to (4.12) and the equations governing the coefficients of Δ^2 in (5.5) include a pair like (5.9) which serve to fix L_1, M_1 and ω_2.

From (5.9) our first result is that

$$q^2[3^{\frac{1}{2}}\omega_1 + i\Delta^{-1}(R-R_{c0})]^2 = 3m^2 - l^2q^2 = 3^{\frac{1}{2}}\lambda(3+q^2) - 2q^2. \tag{5.10}$$

Thus if $\lambda < \lambda_0(q)$, where $\lambda_0(q)$ is given by (4.30), ω_1 must take one or other of the two imaginary values

$$i\omega_{11}, i\omega_{12} = (R-R_{c0})/3^{\frac{1}{2}}\Delta \pm 3^{-\frac{1}{2}}R_{c1}, \tag{5.11}$$

where
$$R_{c1} = -[2q^2 - 3^{\frac{1}{2}}\lambda(3+q^2)]^{\frac{1}{2}}/q. \tag{5.12}$$

Corresponding to these, (5.9) gives

$$\frac{M_0}{L_0} = \mp i\left[\frac{3^{\frac{1}{2}}m + lq}{3^{\frac{1}{2}}m - lq}\right]^{\frac{1}{2}} = \mp i\mu_0, \quad \text{say}. \tag{5.13}$$

Whereas in the theory of Eltayeb (1972) the two modal disturbances corresponding to the chosen values of l, m and R are of equal status, the Taylor shear has removed this degeneracy by selecting particular combinations (5.13) of the two. Recalling that $l > 0$ and $m < 0$, we see that $\lambda = \lambda_0$ coincides with the vanishing of $lq + 3^{\frac{1}{2}}m$ [cf. (4.30)], so that for both the modes (5.11) the amplitude of the positive-m roll is, according (5.13), small compared with that of the negative-m roll. A relationship between these results and the findings of §4 evidently exists.

Still supposing that $\lambda < \lambda_0$, we see from (5.11) that one of the two modes, namely $\omega_1 = \omega_{12}$, is unstable at $R = R_{c0}$ and grows slowly and aperiodically with time, while the other, $\omega_1 = \omega_{11}$, decays. In fact, because of the presence of the Taylor shear, the critical value of R is reduced from $R_{c0} = 3^{\frac{3}{2}}$ to

$$R_c = R_{c0} + \Delta R_{c1} + O(\Delta^2). \tag{5.14}$$

Taking the real parts of expressions such as (5.1), we see that the most unstable mode at $R = R_{c0}$ is given by

$$V_z = L_{02}\cos z[\cos(lx+my) - \mu_0 \sin(lx-my)]\exp(\Delta|R_{c1}|t/3^{\frac{1}{2}}) + O(\Delta), \tag{5.15}$$

where L_{02} is a real constant. When both modes (5.11) are present, we find that to leading order

$$T = (lm^2\mu_0/6\times 3^{\frac{1}{2}})\cos 2lx[L_{02}^2\exp(2\Delta|R_{c1}|t/3^{\frac{1}{2}}) - L_{01}^2\exp(-2\Delta|R_{c1}|t/3^{\frac{1}{2}})]. \tag{5.16}$$

We can choose $L_{01}^2 = L_{02}^2$ so that $T = 0$ at $t = 0$ but it is clear that the different evolution rates of the two modes imply that $|T|$ increases. It follows that F must also evolve. A careful study is needed to investigate its subsequent behaviour but it seems possible that it decays to zero without changing its dependence on x, the variation with t being determined by arguments parallel to those of §4. The final state then consists of a single roll of constant amplitude because as $F \to 0$ the critical value of R returns to R_{c0}.

On the other hand, when $\lambda > \lambda_0(q)$, ω_{11} and ω_{12} are real but of opposite signs, so that $R_{c1} = 0$ and the presence of the Taylor shear does not alter the critical value of R to order Δ. There is, however, a slow phase change in the two solutions, which means that they cannot be combined to form a simple roll of permanent form. Thus, considering one dependent variable V_z as an illustration, we have

$$V_z = L_{01} \cos z[\cos(lx+my \pm \tilde{\omega}_1 \Delta t) \mp \mu_0 \cos(lx-my \pm \tilde{\omega}_1 \Delta t)], \tag{5.17}$$

where

$$3^{\frac{1}{2}}\tilde{\omega}_1 = 3^{\frac{1}{4}}(3\epsilon q^2)^{\frac{1}{2}} (\lambda - \lambda_0)^{\frac{1}{2}}. \tag{5.18}$$

It is clearly possible to choose $L_{01}^2 = L_{02}^2$, so that the disturbance initially consists of a simple roll [(3.1) or (3.2)], but as time increases the other roll must come into prominence and indeed we see that, formally, a complete exchange takes place when $\tilde{\omega}_1 \Delta t = \frac{1}{2}\pi$. As when $\lambda < \lambda_0$ the appearance of the other roll leads to a non-zero value of T; indeed using (5.7) and $L_{01} = L_{02}$, we have

$$T = 3^{-\frac{3}{4}} \mu_0 lm^2 L_{01}^2 \sin 2lx \sin 2\alpha\tilde{\omega}_1 t, \tag{5.19}$$

showing that F must again evolve with time. Now, however, our studies in §4 show that if $\lambda_0 < \lambda < \lambda_1$ the negative-m roll is unstable and so the evolution of F ends when the disturbance either consists of a simple positive-m roll, or takes on a more complicated form in which each of the two marginally stable rolls are of comparable strength or is even an irregular motion. If $\lambda_1 < \lambda < 2/3^{\frac{1}{2}}$ both simple rolls are unstable and so the first of these possibilities must then be excluded. The discussion of this evolutionary process is rendered more difficult by the form of (5.19), in which T is proportional to $\sin 2lx$ rather than $\cos 2lx$ as is the case in §4. The implications of this change are not yet fully understood.

6. Conclusions

In this paper we have considered a neutral state of equilibrium when $q < 2$, so that stability is about to be lost by 'the exchange of stabilities'. Within the range $0 < \lambda < 2/3^{\frac{1}{2}}$, in which the linear stability problem is degenerate, we have located two values,

$$\lambda_0 = 2q^2/3^{\frac{1}{2}}(q^2+3), \quad \lambda_1 = 2^{-1} + 3^{-\frac{1}{2}} \quad (> \lambda_0), \tag{6.1}$$

that appear to enjoy a particular significance.

In §4, we have examined the evolution of two rolls, with initial amplitudes A_1 and A_2 ($\ll A_1$), i.e. we have examined the linear stability of an oblique roll (A_1) of RS with respect to a perturbation by the other oblique roll (A_2). It was found that this stability depends on whether the A_1 roll is supposed to be the positive-m mode, i.e. the one whose eigenfunction is proportional to $\exp[i(lx+my)]$, where $lm > 0$, or whether it is the negative-m roll, for which $lm < 0$. Both possibilities are linearly stable if $\lambda < \lambda_0$. If λ is increased through λ_0, the negative-m roll becomes unstable, and if λ is increased still further, the positive-m roll follows suit. The value of λ marking the onset of this instability depends of course on q, but also in a complicated way on p. If $p \ll 1$, a value of λ exists, namely $\lambda = \lambda_1$, at which the real part of two conjugate complex roots of the governing septic equation defined by (4.19) changes sign. For simplicity of discussion, we shall regard $\lambda = \lambda_1$ as marking the transition from stability to instability in all circumstances. [Alternatively the reader may consider that λ_1 is not given by (4.29) but is a function of q and p that is unknown, except for $q = 1$, at which it happens to coincide for all p with that defined in (4.29).] In the range $\lambda_0 < \lambda < \lambda_1$ the negative-m roll only is unstable; in $\lambda_1 < \lambda < 2/3^{\frac{1}{2}}$ both types of roll are unstable.

There is one firm conclusion that can be drawn from these results: the single-roll solutions of the type studied in RS will not generally arise if $\lambda_1 < \lambda < 2/3^{\frac{1}{2}}$. For, if one roll were to evanesce, its amplitude A_2 would ultimately become small compared with A_1 and according to the theory of § 4 it would start to grow once more. It is very probable that both rolls will persist indefinitely with comparable amplitudes and with an associated Taylor shear more complicated than that assumed in (4.5). As a result, however, the rolls themselves will be modified and not have the simple forms of (3.1) and (3.2). Alternatively the laminar flow tacitly assumed throughout this paper may be lost and a transition to turbulence take place. In the range $\lambda_0 < \lambda < \lambda_1$ the same possibilities exist but in addition the system may well settle down to a single roll of positive m with a zero associated Taylor shear. Indeed if the other roll is initially weak this outcome is certain. If $\lambda < \lambda_0$, we again have the same possibilities as when $\lambda_0 < \lambda < \lambda_1$ and the remarks made about the positive-m roll also apply to the negative-m roll since both are then stable to small disturbances.

A possibly significant difference in the modes of instability arose at $\lambda = \lambda_0$ and $\lambda = \lambda_1$. If λ is increased through λ_0, a single real solution of the septic equation defined by (4.19) for the growth rates s of the normal modes passes from negative to positive values. This value of s is $O(1)$, by which we mean that it is proportional to $\lambda - \lambda_0$ with an $O(1)$ constant of proportionality, irrespective of the value of $\delta/\epsilon q$. If $\delta/\epsilon q$ is small, this root can be found to leading order by solving the quintic equation obtained by formally setting $\delta/\epsilon q = 0$ in the septic equation. In other words, it can be obtained from the procedure laid down by Taylor (1963). If λ is increased sufficiently, the positive-m roll becomes unstable, but in a way that cannot be understood by Taylor's method. If $\delta/\epsilon q$ is small, the instability takes the form of a pseudo-Alfvén wave (with a phase velocity proportional to the *perturbed* magnetic field) whose amplitude increases on the $O(1)$ time scale.

The excitation of the Taylor shear by the double-roll solutions played an essential part in the evolutionary process just described. In § 5 we inquired how a small Taylor shear, constant in time, will affect the linear stability problem. As in RS, the theory ignored the inertial terms, and no significance for λ_1 was therefore discovered. Once again, however, λ_0 enjoyed a special significance. It was found that the small Taylor shear removes the degeneracy of the problem considered by RS, and selects two particular values for A_1/A_2. For $\lambda < \lambda_0$, the critical Rayleigh number for the onset of convection in these coupled roll solutions is different, one coupled roll being more readily excited to convection, and the other less readily, compared with the RS situation. For fixed Rayleigh number, the mode corresponding to one value of A_1/A_2 grows more rapidly than the other, again leading to a violation of Taylor's condition.

In discussing the possible geophysical relevance of our work we should make a number of points clear at the outset. First, the value of q is extremely small in the earth's core ($q \doteqdot 3 \times 10^{-6}$). Although our present theory (valid for $q < 2$) is applicable, additional geophysically interesting information could be expected from a theory that treated q as small from the outset. Second, the model can claim relevance to the earth only if the core is in an approximately isentropic state. There have been suggestions that the core is strongly and stably stratified. Vertical motions in such a core would be greatly inhibited, and it is known that purely toroidal motions are incapable of sustaining fields by dynamo action (e.g. Roberts 1967, p. 82). Our firm opinion is in fact that the existence of a geomagnetic field demonstrates that the prevailing temperature gradient in the core cannot be grossly sub-adiabatic. Third, the theory is concerned with values of R near $3^{\frac{3}{2}}$, whereas R is likely to be very large. In a private communication Dr Braginskiĭ has suggested to us that R might be as large as 10^7. Caution must therefore be exercised in applying the theory to the earth. Fourth, the theory is for plane boundaries, and an extension to spherical boundaries is needed for geophysical applications. Dr Soward has privately suggested to us that the presence of spherical boundaries might imply restrictions on the properties of the unstable waves and in particular that it might then be impossible to generate a single roll under any circumstances. This is certainly true for the planetary waves studied by Margules (1893) and Haurwitz (1940). Further analysis is clearly necessary to decide this important point, but we note that, unlike the convection rolls discussed here, variations with respect to z are neglected in planetary theory. Further, in studies of inertial waves (e.g. Stewartson & Walton 1975) in which there is a variation in z, the mechanism preventing the existence of single rolls depends on the presence of critical circles and, in turn, these depend on retaining the term $\delta^2 \partial \mathrm{V}/\partial t$ in (2.2d), which is neglected in the Eltayeb theory.

The close coincidence of the geographic and geomagnetic axes when averaged over the recent geological past indicates the importance of Coriolis forces in core dynamics. This may be confirmed if we note that the Rossby number and Ekman number are both small in the core. If we take as the characteristic velocity u the speed of westward drift ($\doteqdot 10^{-4}$ m/s) and as the characteristic length the core radius a, we see that the Rossby number $u/2\Omega a$ is about 2×10^{-7}. The kinematic viscosity of the core possibly lies between $10^{-7}\,\mathrm{m^2/s}$ and $10^2\,\mathrm{m/s}$, which leads to Ekman numbers $\nu/2\Omega a^2$ in the range 10^{-16}–10^{-7}.

The near neutral buoyancy of the core and the importance of Coriolis forces give a special significance to the geostrophic component of the core motion. This flow is along lines of latitude and is constant on each cylindrical shell drawn about the geographical axis and contained in the core. It is unique in being completely uninfluenced by rotation, so that small forces that are inhibited by Coriolis forces from contributing to other components of the flow can create disproportionately large geostrophic motions. In analogy with concepts in the atmospheric sciences, transport of mean angular momentum by rising and falling motions in the core may be expected to excite significant geostrophic flows, and it is tempting to identify these at least in part with the observed westward drift, and to assume that the core motions are predominantly geostrophic and of magnitude comparable with the observed westward drift. The corresponding magnetic Reynolds number $R_m = ua\mu\sigma$ of this flow is large ($R_m \doteqdot 100$).

The existence of a geostrophic shear ω associated with a large magnetic Reynolds number implies that the toroidal field in the core is large, about R_m times the poloidal field. Extrapolation of the observed (poloidal) field at the surface of the earth back to the core surface suggests that the poloidal field in the core is about 4 gauss, so that the toroidal field there may be of order 400 gauss. The corresponding value of λ is 0·003. It should be realized however that there is considerable uncertainty here and that, on the basis of different arguments, values of the mean toroidal field between 100 gauss (Hide 1966) and 740 gauss (Braginskiĭ 1964*b*) have been suggested at various times, giving respectively $\lambda = 0{\cdot}05$ and $\lambda = 0{\cdot}0009$, both of which are large compared with $\lambda_0 = 3\times 10^{-12}$ for $q = 3\times 10^{-6}$.

To complete the dynamo cycle, it is necessary to use the toroidal field to recreate the poloidal field. This cannot be achieved directly since Cowling's theory forbids axisymmetric dynamo fields. The next simplest concept is the two-stage Parker (1955) process, in which a motion asymmetric with respect to the longitude ϕ creates a field with like asymmetry. The inductive interactions of this field and motion regenerate axisymmetric fields. This mechanism, in a different guise, was called by Steenbeck, Krause & Rädler (1966) the 'α-effect'. In the case of the earth, it appears to create an axisymmetric toroidal field that is negligible compared with that produced by the ω-shear described above. Thus, only the α-production of poloidal field need be considered. The resulting model is sometimes called an $\alpha\omega$-dynamo since it functions through the product of α-effect and ω-shear. The concept of an $\alpha\omega$-dynamo requires special care in the present context since an exactly geostrophic flow cannot regenerate field in the face of Taylor's constraint (Childress 1969). Also, the α-process we have in mind is not that of the turbulent dynamo of Steenbeck *et al.* (1966), but is the wave interaction mechanism suggested by Braginskiĭ (1964*c*), which we shall now consider.

The $\alpha\omega$-dynamo just defined gives a new significance to the observed asymmetries of the earth's field. They must be a manifestation of a large-scale instability that passes energy (possibly from the gravitational source provided by a slightly top-heavy density distribution, thermally or non-thermally created) to the large axisymmetric scales of the magnetic field. And reasons have been found (e.g. Hide & Stewartson 1972) why the preferred state of instability (i.e. the mode most readily excited to convection) should be asymmetric rather than axisymmetric. This consistency with Cowling's theorem is reassuring, but a fundamental difficulty remains. It was shown by Braginskiĭ (1964*a*) that no α-production of polodial field could arise if the waves were separable in $\phi - \tilde{\omega}t$, where $\tilde{\omega}$ is their angular velocity about the symmetry axis; the regenerative cycle would fail to create poloidal field if, for example, the radial velocity associated with the wave could be written as $W(r,z)\cos(\phi - \tilde{\omega}t)$, where (r, ϕ, z) are cylindrical co-ordinates (see also D. I. Black, reported by Gubbins 1973). The inference is that the eigenvalue problem determining the onset of the waves must be degenerate. It must be one of the objectives of any dynamical theory to discover how such degeneracies can occur, and to confirm that they can create poloidal field. The harder task of ensuring that these fields are strong enough to sustain the dynamo may be regarded as a more distant objective of the theory.

Another aim should be mentioned. There is a common belief, whose plausibility can be confirmed from the estimate made above, that the Coriolis forces in the core are of the same order as the Lorentz forces, i.e. $2\rho\Omega u$ is roughly of the same order as $B_0^2/\mu a$, where B_0 and u are a typical toroidal field strength and toroidal velocity. (As a corollary, the typical Alfvén velocity $V\,[= B_0/(\mu\rho)^{\frac{1}{2}}]$ based on the toroidal field strength would be large compared with u, and the magnetic energy density would be large compared with the kinetic energy density of the motions relative to the rotating frame.) We have also seen that the condition for dynamo action is that the magnetic Reynolds number R_m should to geophysical accuracy be of order unity. Thus, the approximate equality of Coriolis and Lorentz forces implies that λ should be of order one in the core. The dynamical reason for this should be sought.

We now consider what light our work throws on these questions. For reasons of analytical expediency, we have assumed a planar geometry, in which the y direction corresponds to longitude and the z direction to the radial direction in a spherical case. Our theory requires that the two parameters δ and ϵ should both be small compared with unity. Substituting our estimates of § 1 into (2.3) we see that δ is approximately 10^{-8}. It is more difficult to estimate ϵ. If we took $B_0\hat{\mathbf{y}}$ in (2.1) to be the toroidal field and $\mu\sigma\kappa\epsilon B_0\mathbf{B}$ to be the non-dipole field, we should find that $\epsilon \simeq 300$. It should not be forgotten however that ours is a $q = O(1)$ theory, and that the criterion for the validity of the $q \to 0$ theory we advocated earlier is unlikely to be as simple as $\epsilon \ll 1$. Further, the study of the fields when $\epsilon \ll 1$ can be regarded as a first step in the regenerative process which ultimately leads to larger values of ϵ. For this reason we continue to examine the geophysical context, and note that $\delta/\epsilon q$ is about 10^{-5} according to values given above.

Turning now to the difficulty of α-production by wave instabilities (Braginskiĭ 1964*c*), we see that the required degeneracy of the eigenvalue problem can arise in our model provided that $\lambda < 2/3^{\frac{1}{2}}$. Moreover, it can be shown that an α-effect exists, the double roll being associated with an electric current that is independent of y and parallel to $\mathbf{B}_0$. The combined α-effect and ω-shear are not, however, sufficient for a regenerative $\alpha\omega$-dynamo in our model. Assuming that this deficiency can be overcome in a more elaborate model, we examine possible implications of our work in dynamo theory.

If $\lambda > 2/3^{\frac{1}{2}}$, the preferred mode of instability is a single roll transverse to $\mathbf{B}_0$, and the absence of any degeneracy excludes dynamo action. To be self-sustaining, the magnetic field must be perturbed past this threshold. We have found indications, described above, that as λ decreases one roll has more permanence than the other and is excited to a greater amplitude. This suggests that the required α-effect will become progressively less efficient as the dynamo strengthens the toroidal field $\mathbf{B}_0$, an opinion confirmed by the linear stability of both rolls for $\lambda < \lambda_0$. This strongly suggests the existence of a dynamical mechanism of selfadjustment that might limit λ to values of order unity as required by the geophysical application (see above).

Another pertinent question concerns the use of Taylor's procedure in theoretical studies of core magnetohydrodynamics. Since some of our more interesting results concerning the behaviour of instabilities of the quasi-Alfvén type are filtered out from Taylor's approximation, we have some reservations about the indiscriminate application of Taylor's method to core motions. We believe it is necessary to retain the inertial terms in the equation of geostrophic flow if these doubts are to be dispelled.

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